An Accelerated Semi-Analytical Coupled Line Gauss-Seidel Smoother (ASA-CLGS) for multigrid solution of incompressible Navier-Stokes equations

Yu. Feldman and A. Gelfgat

School of Mechanical Engineering
Faculty of Engineering
Tel-Aviv University

9th European Multigrid Conference EMG2008

Outline

- **Pressure-velocity coupled formulation of the Navier-Stokes equations**
- **The Multigrid Approach**
- **Numerical Technique**
- **Analytical Solution Accelerated (ASA) smoother**
- **Comparison with existing benchmark solutions**
- **Conclusions**

Incompressible Navier Stokes Equations

Continuity - $$\nabla \cdot \boldsymbol{u} = 0$$

Momentum- $$\frac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u} \cdot \nabla\right)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \boldsymbol{u}$$

- No time derivative of pressure
- No boundary conditions for pressure

Incompressible Navier Stokes Equations (Cont.)

Projection methods for pressure-velocity decoupling

- ✓ Good numerical robustness
- ✓ Low memory consumption
- ✘ Slow rate of numerical convergence
- ✘ Not physical pressure field
- ✘ Not applicable for flow–structure interaction problems

Pressure–velocity coupled approach

- ✓ Good numerical convergence
- ✓ The "most natural" way to solve N-S equations
- ✓ The obtained pressure is physical
- ✘ Large memory consumption
- ✘ Not as numerically robust as pressure projection methods

Multigrid Approach-the Optimal Choice

- ✓Low memory and CPU time consumption, *O(N)*
- ✓ Pressure-velocity coupling can be utilized
- ✓ Easily parallelized (by MPI or Open MP tools)
- ✘ Non-constant convergence rate
- ✘ Sophisticated programming is needed

Discretization in time and space

Second order backward differentiation - $$\frac{\partial f^{n+1}}{\partial t} = \frac{3f^{n+1} - 4f^{n} + f^{n-1}}{2\Delta t} + O\left(\Delta t^2\right)$$

Energy - $$\left(a^{\theta}_{(i,j,k)} - \frac{3}{2\Delta\tau} \right) \theta^{n+1}_{(i,j,k)} + \sum_{i,j,k} a^{\theta}_{i,j,k}\, \theta^{n+1}_{i,j,k} = RHP^{n}_{\theta}$$

Temperature – velocity decoupling

Continuity - $$\frac{\left(u^{n+1}_{(i,j,k)} - u^{n+1}_{(i-1,j,k)}\right)}{Hx(i-1)} + \frac{\left(v^{n+1}_{(i,j,k)} - v^{n+1}_{(i,j-1,k)}\right)}{Hy(j-1)} + \frac{\left(w^{n+1}_{(i,j,k)} - w^{n+1}_{(i,j,k-1)}\right)}{Hz(k-1)} = 0$$

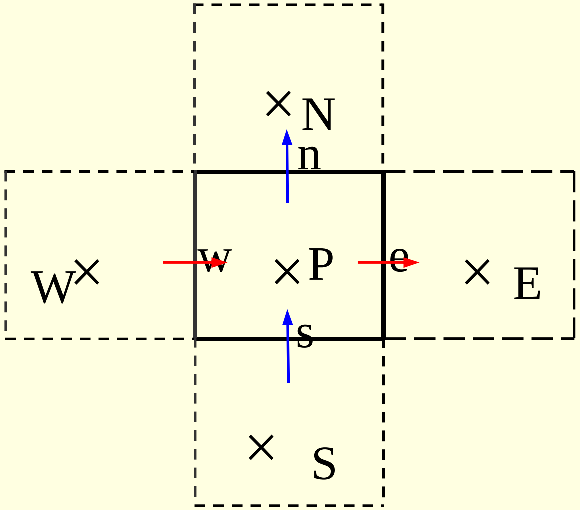
Linearized Navier-Stokes equation; l.h.s. = Stokes operator

Momentum- $$\left(a^{\mathbf{u}}_{(i,j,k)} - \frac{3}{2\Delta\tau} \right) \boldsymbol{u}^{n+1}_{(i,j,k)} + \sum_{(i,j,k)} a^{u}_{(i,j,k)}\, \boldsymbol{u}^{n+1}_{(i,j,k)} - \nabla p^{(n+1)} = RHP^{n}_{\mathbf{u}}$$

Conservative second order control volume method

Symmetrical coupled Gauss-Seidel smoothing operator (SCGS)

S.P. Vanka (1985) – analytical solution for a ***single*** finite volume



$$(u,v)^{\text{new}} = (u,v)^{\text{old}} + r_{(\text{u,v})}(u,v)'$$

$$p^{\text{new}} = p^{\text{old}} + r_{\text{p}}p'$$

$$A_1 = a_e^u - \frac{3}{2\Delta\tau} \qquad A_3 = a_w^u - \frac{3}{2\Delta\tau}$$

$$A_5 = a_n^u - \frac{3}{2\Delta\tau} \qquad A_9 = a_s^u - \frac{3}{2\Delta\tau}$$

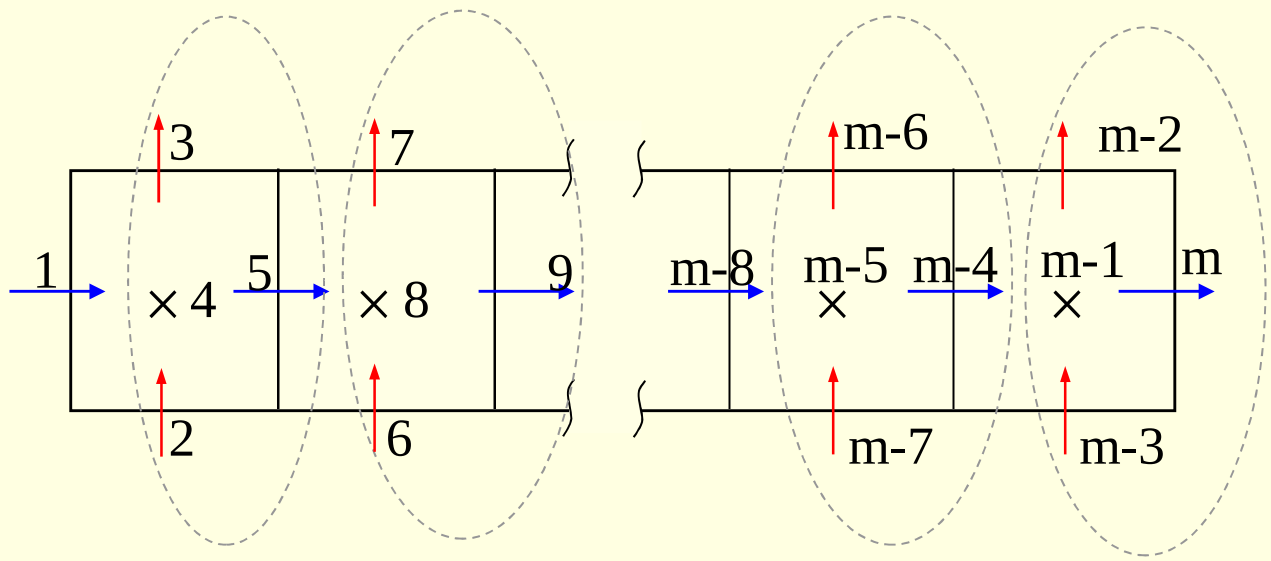
$$\begin{bmatrix} A_1 & 0 & 0 & 0 & A_2 \\ 0 & A_3 & 0 & 0 & A_4 \\ 0 & 0 & A_5 & 0 & A_6 \\ 0 & 0 & 0 & A_9 & A_{10} \\ A_7 & -A_7 & A_8 & -A_8 & 0 \end{bmatrix} \times \begin{bmatrix} u'_{\text{e}} \\ u'_{\text{w}} \\ v'_{\text{n}} \\ v'_{\text{s}} \\ p'_{\text{P}} \end{bmatrix} = \begin{bmatrix} R_{\text{ue}} \\ R_{\text{uw}} \\ R_{\text{vn}} \\ R_{\text{vs}} \\ R_{\text{cp}} \end{bmatrix}$$

for the Stokes operator and a constant time step A_1, A_3, A_5, A_9 are constants

Accelerated coupled line Gauss-Seidel smoother (ASA-CLGS) -2D

Zeng and Wesseling (1993) – CLGS: Horizontal (vertical) sweeping *with* horizontally (vertically) adjacent pressure linkage

Feldman and Gelfgat (2008) – ASA-CLGS:Horizontal (vertical) sweeping *without* horizontally (vertically) adjacent pressure linkage



Accelerated coupled line Gauss-Seidel smoother (ASA-CLGS) -2D, (Cont)

Zeng and Wesseling (1993) – CLGS:

$$\cdots$$

$$A^{(x)}_{i+1/2,j}u'_{i+1/2,j}+B^{(x)}_{i+1/2,j}p'_{i,j}=R^{(x)}_{i+1/2,j}$$

$$A^{(x)}_{i-1/2,j}u'_{i-1/2,j}-B^{(x)}_{i-1/2,j}p'_{i,j}=R^{(x)}_{i-1/2,j}$$

$$A^{(y)}_{i,j+1/2}v'_{ij+1/2}-B^{(y)}_{i,j+1/2}\left(p'_{i,j+1}-p'_{i,j}\right)=R^{(y)}_{i,j+1/2}$$

$$A^{(x)}_{i,j}\left(u'_{i+1/2,j}-u'_{i-1/2,j}\right)+A^{(y)}_{i,j}\left(v'_{i,j+1/2}-v'_{i,j-1/2}\right)=0$$

$$\cdots$$

Feldman and Gelfgat (2008) – ASA-CLGS:

$$\cdots$$

$$A^{(x)}_{i+1/2,j}u'_{i+1/2,j}+B^{(x)}_{i+1/2,j}p'_{i,j}=R^{(x)}_{i+1/2,j}$$

$$A^{(x)}_{i-1/2,j}u'_{i-1/2,j}-B^{(x)}_{i-1/2,j}p'_{i,j}=R^{(x)}_{i-1/2,j}$$

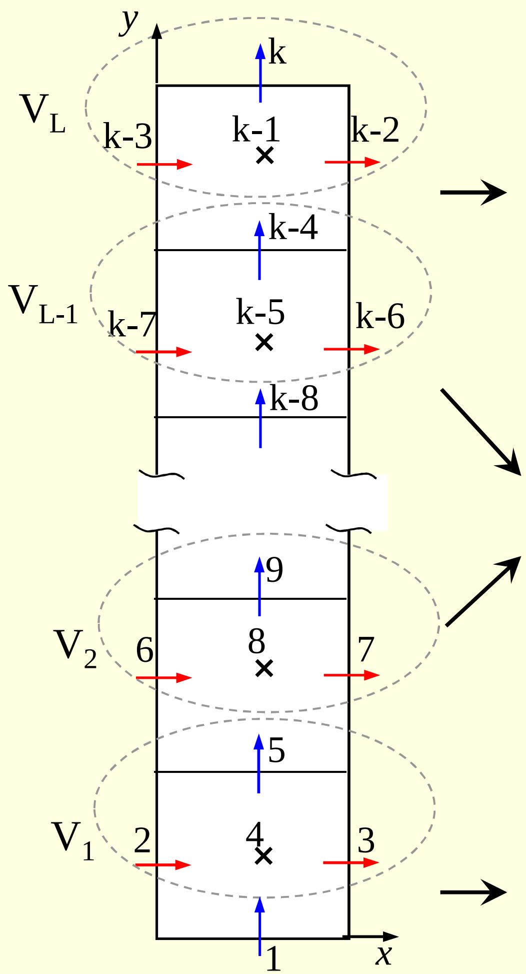
$$A^{(y)}_{i,j+1/2}v'_{ij+1/2}+B^{(y)}_{i,j+1/2}p'_{i,j}=\widetilde{R}^{(y)}_{i,j+1/2}$$

$$A^{(x)}_{i,j}\left(u'_{i+1/2,j}-u'_{i-1/2,j}\right)+A^{(y)}_{i,j}\left(v'_{i,j+1/2}-v'_{i,j-1/2}\right)=0$$

$$\cdots$$

where $\widetilde{R}^{(y)}_{i,j+1/2}=R^{(y)}_{i,j+1/2}+B^{(y)}_{i,j+1/2}p'_{i,j+1}$

A schematic description of ASA-CLGS smoother.



$$\begin{bmatrix} A_1^L & 0 & 0 & 0 & A_2^L \\ 0 & A_3^L & 0 & 0 & A_4^L \\ 0 & 0 & A_5^L & 0 & A_6^L \\ 0 & 0 & 0 & 1 & 0 \\ A_7^L - A_7^L & A_8^L - A_8^L & & & 0 \end{bmatrix} \times \begin{bmatrix} u'_{k-2} \\ u'_{k-3} \\ v'_{k} \\ v'_{k-4} \\ p'_{k-1} \end{bmatrix} = \begin{bmatrix} R_{k-2} \\ R_{k-3} \\ R_{k} \\ R_{k-4} \\ R_{k-1} \end{bmatrix} \Rightarrow \begin{array}{l} u'_{k-2} = \alpha^L_{k-2} p'_{k-1} + \beta^L_{k-2} \\ u'_{k-3} = \alpha^L_{k-3} p'_{k-1} + \beta^L_{k-3} \\ v'_{k} = \textit{from b.c.} \\ v'_{k-4} \textit{ from volume } L-1 \\ p'_{k-1} = \alpha^L_{k-1} v'_{k-4} + \beta^L_{k-1} \end{array}$$

$$\begin{bmatrix} A_1^I & 0 & 0 & 0 & A_2^I \\ 0 & A_3^I & 0 & 0 & A_4^I \\ 0 & 0 & A_5^I & 0 & A_6^I \\ 0 & 0 & 0 & 1 & 0 \\ A_7^I - A_7^I & A_8^I - A_8^I & & & 0 \end{bmatrix} \times \begin{bmatrix} u'_{m-2} \\ u'_{m-3} \\ v'_{m} \\ v'_{m-4} \\ p'_{m-1} \end{bmatrix} = \begin{bmatrix} R_{m-2} \\ R_{m-3} \\ R_{m} \\ R_{m-4} \\ R_{m-1} \end{bmatrix} \Rightarrow \begin{array}{l} u'_{m-2} = \alpha^L_{m-2} p'_{m-1} + \beta^L_{m-2} \\ u'_{m-3} = \alpha^L_{m-3} p'_{m-1} + \beta^L_{m-3} \\ v'_{m} = \alpha^L_{m} p'_{m-1} + \beta^L_{m-4} \\ v'_{m-4} \textit{ from volume } M-1 \\ p'_{m-1} = \alpha^L_{m-1} v'_{m-4} + \beta^L_{m-1} \end{array}$$

$$\begin{bmatrix} A_1^1 & 0 & 0 & 0 & A_2^1 \\ 0 & A_3^1 & 0 & 0 & A_4^1 \\ 0 & 0 & A_5^1 & 0 & A_6^1 \\ 0 & 0 & 0 & 1 & 0 \\ A_7^1 - A_7^1 & A_8^1 - A_8^1 & & & 0 \end{bmatrix} \times \begin{bmatrix} u'_{3} \\ u'_{2} \\ v'_{5} \\ v'_{1} \\ p'_{4} \end{bmatrix} = \begin{bmatrix} R_{3} \\ R_{2} \\ R_{5} \\ R_{1} \\ R_{4} \end{bmatrix} \Rightarrow \begin{array}{l} u'_{3} = \alpha^L_{3} p'_{4} + \beta^L_{3} \\ u'_{2} = \alpha^L_{2} p'_{4} + \beta^L_{2} \\ v'_{5} = \alpha^L_{5} p'_{4} + \beta^L_{5} \\ v'_{1} \textit{ from b.c.} \\ p'_{4} = \alpha^L_{4} v'_{1} + \beta^L_{4} \end{array}$$

CLGS and ASA-CLGS Efficiency Estimation for 2D

Zeng and Wesseling (1993) – CLGS:

Block 3-diagonal matrix or 7-diagonal matrix

block-LU decomposition

$\approx O(15M)$

Feldman and Gelfgat (2008) – ASA-CLGS (6-Diagonal Matrix)

$$p'_{k-1} = (c_1^L v'_{k-4} + R_{k-1}^L + \sum_{i=2}^{4} c_i^L R_{k-i}^L)/c_5^L$$

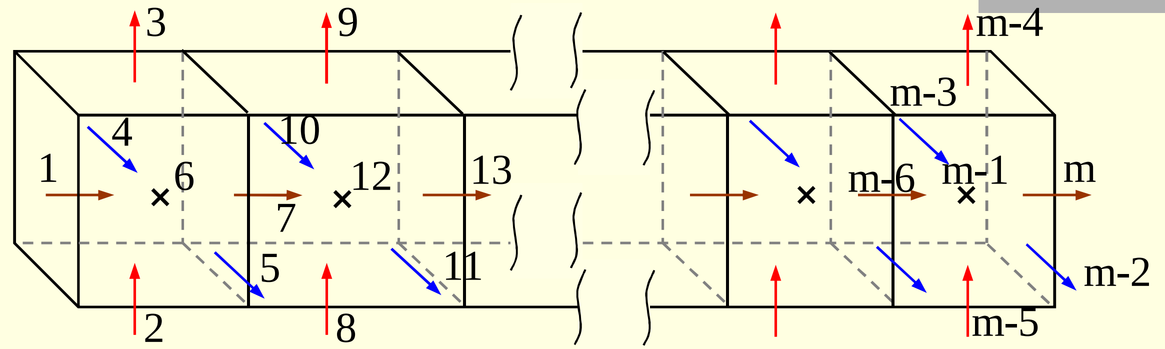
$$\begin{bmatrix} v'_5 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} c_6^1 \\ c_7^1 \\ c_8^1 \end{bmatrix} \times p'_4 + \begin{bmatrix} c_9^1 R_5^L \\ c_{10}^1 R_2^L \\ c_{11}^1 R_3^L \end{bmatrix}$$

$\approx O(5M)$

Thomas Algorithm (3-Diagonal Matrix)

$\approx O(5M)$

ASA-CLGS -Efficiency Estimation for 3D



$$p'_p = \left(c_1^I w'_d + R_p^I + c_2^I R_e^I + c_3^I R_w^I + c_4^I R_n^I + c_5^I R_5^I + c_6^I R_d^I\right) / c_7^I$$

$$\begin{bmatrix} w'_d \\ v'_s \\ v'_n \\ u'_w \\ u'_e \end{bmatrix} = \begin{bmatrix} c_8^I \\ c_9^I \\ c_{10}^I \\ c_{11}^I \\ c_{12}^I \end{bmatrix} \times p'_p + \begin{bmatrix} c_{13}^I R_d^I \\ c_{14}^I R_s^I \\ c_{15}^I R_n^I \\ c_{16}^I R_w^I \\ c_{17}^I R_e^I \end{bmatrix}$$

6 corrections for a single volume result in 17 multiplications and divisions and 11 summations

$$\approx O(5M)$$

Advantages of ASA-CLGS Approach

Zeng and Wesseling (CLGS, 1993) | Feldman and Gelfgat (ASA-CLGS, 2008)

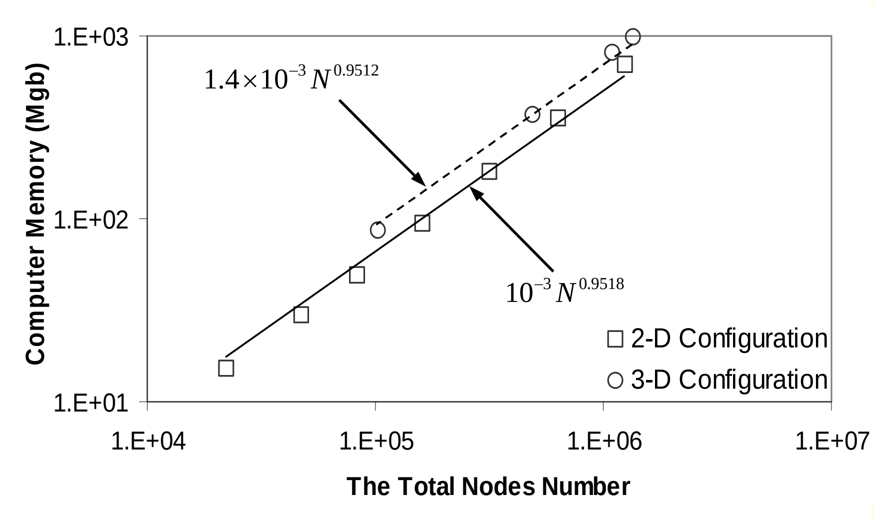
☑ **Still effective for stretched grids.**

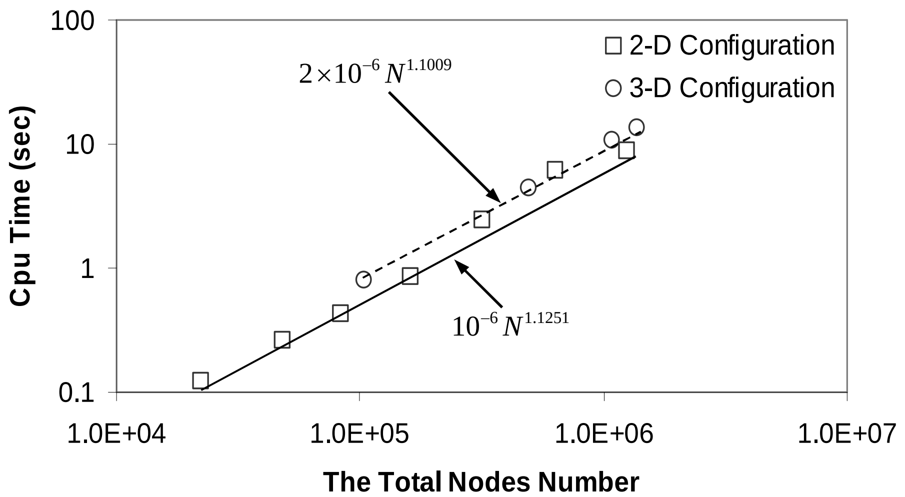
☑ **Still effective for flows with a dominating direction**

- ✘ **Block three-diagonal system is to be solved numerically.**
- ✘ **Increasing amount of arithmetic operations when passing from 2D to 3D geometry**

- ☑ **There exists an analytical solution for the entire corrections row (column).**
- ☑ **Only *O*(5M) operations are needed to obtain the entire row (column) corrections per one sweep for both 2D and 3D geometries**

The Multigrid Characteristics



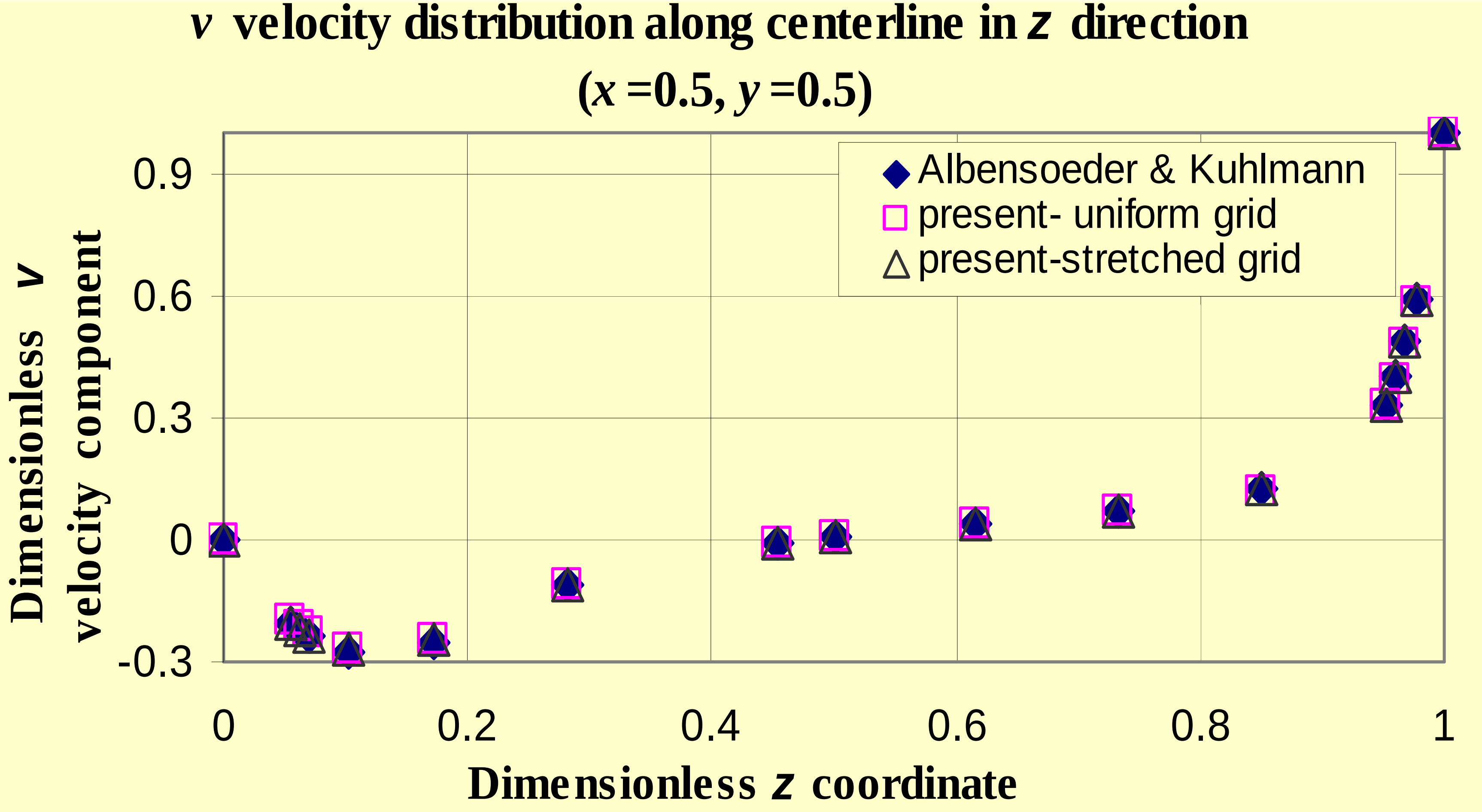


- ☑ Approximately *O*(N) of the CPU memory and time consumption for both 2D and 3D configurations

Cubic lid- driven cavity, grid resolution 103^3

Comparison with Albensoeder & Kuhlmann, 2005.

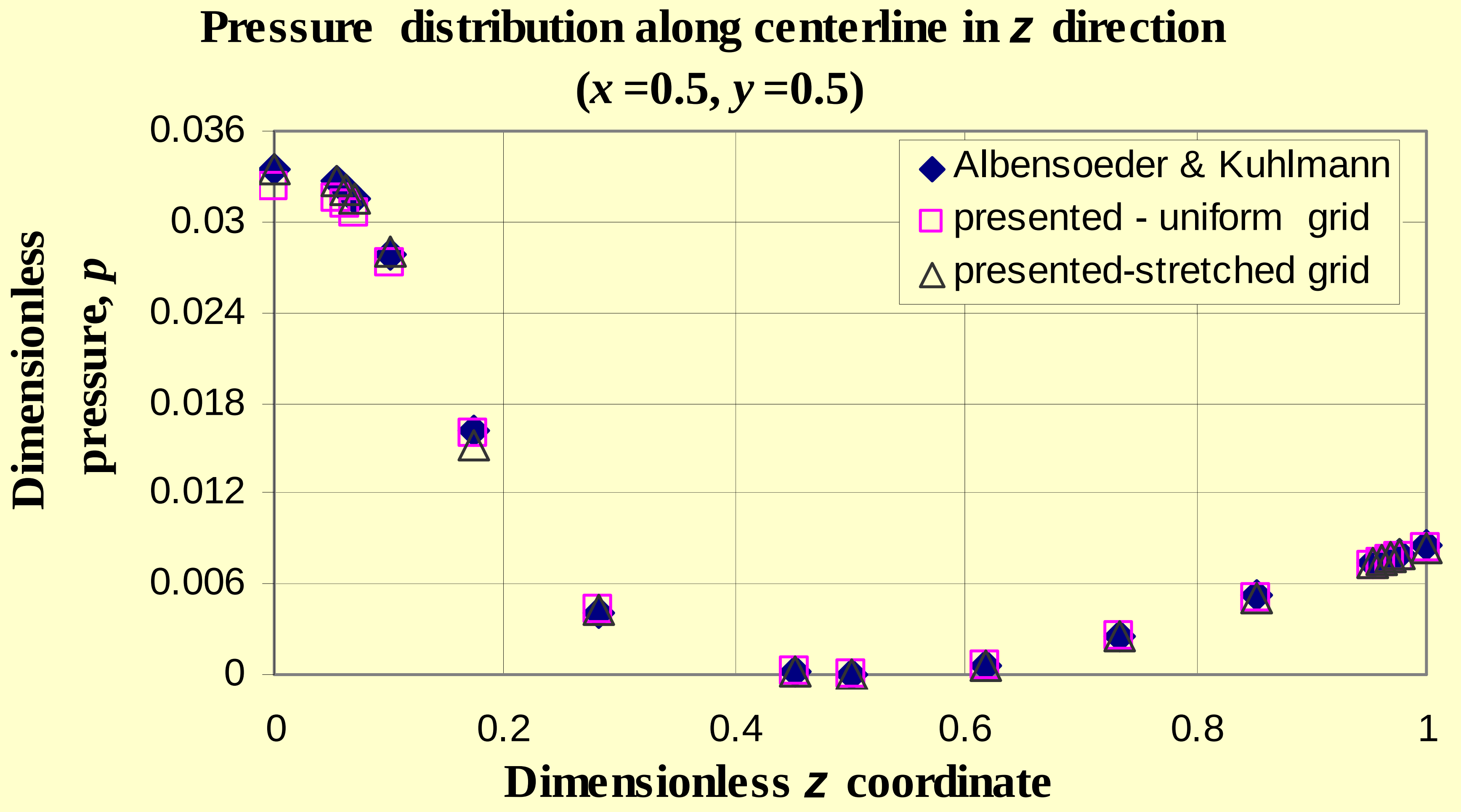
flow at Re= 1000



Cubic lid-driven cavity, grid resolution 103^3 (cont)

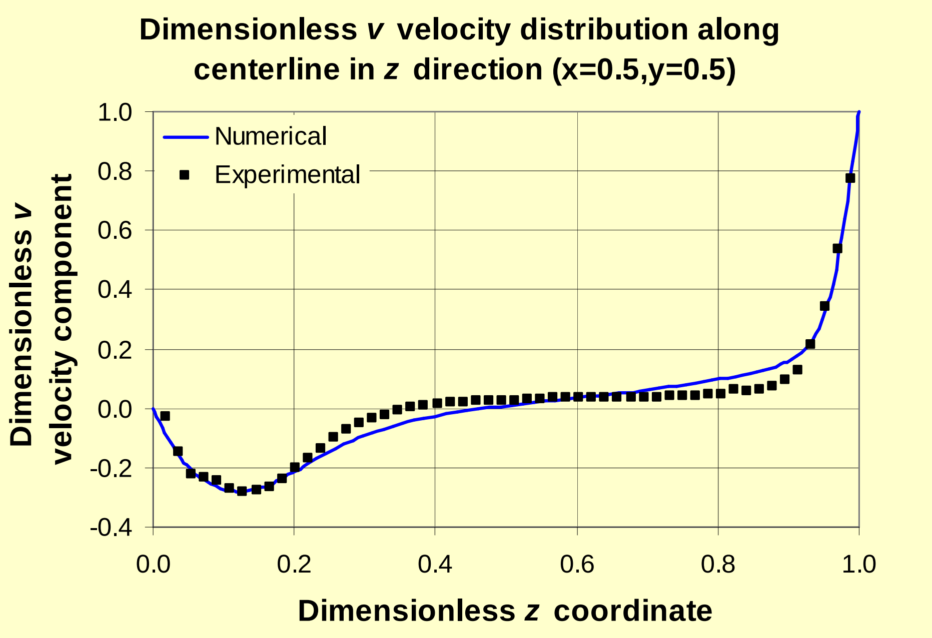
Comparison with Albensoeder & Kuhlmann, 2005.

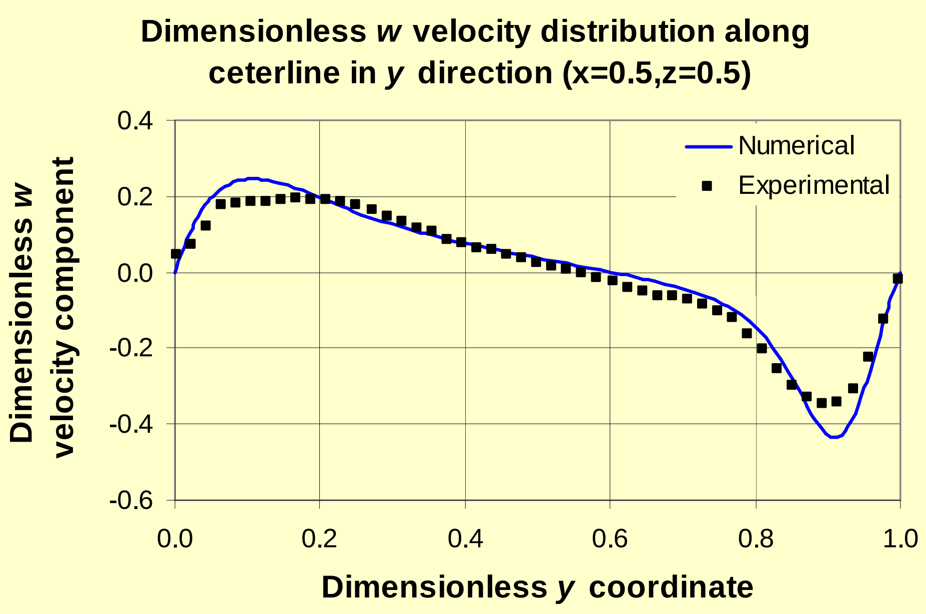
flow at Re= 1000



Cubic lid-driven cavity, grid resolution 103^3 (Cont.2)

Comparison with experiments of A. Liberzon, 2008. flow at Re= 1000

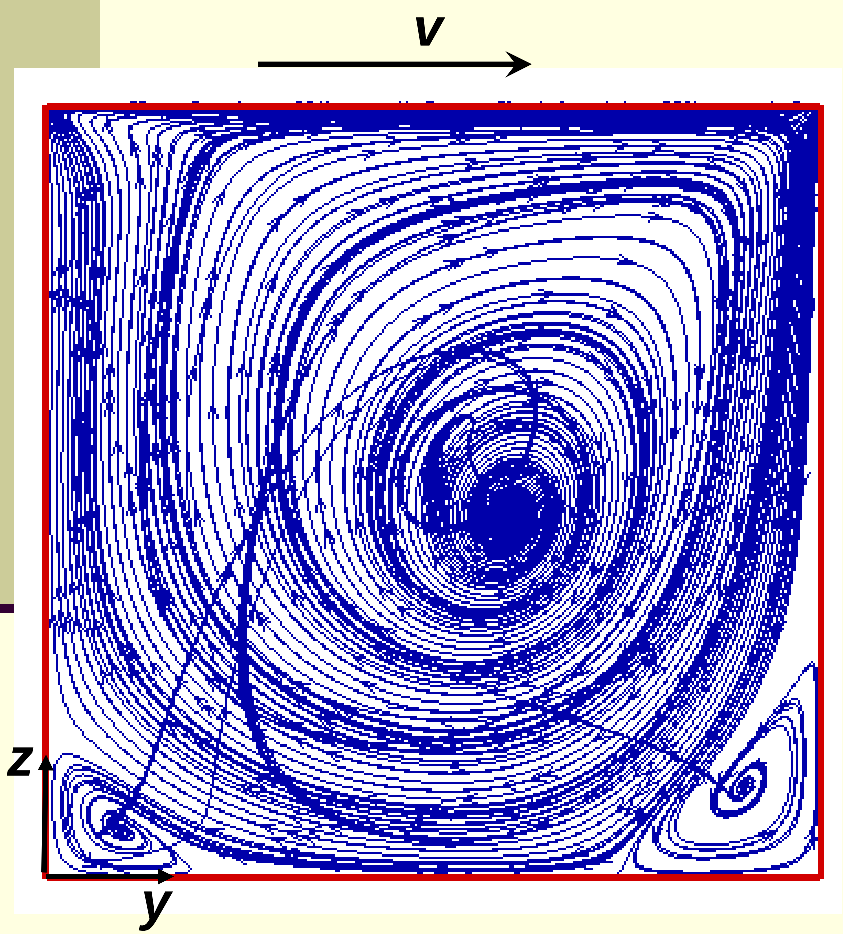


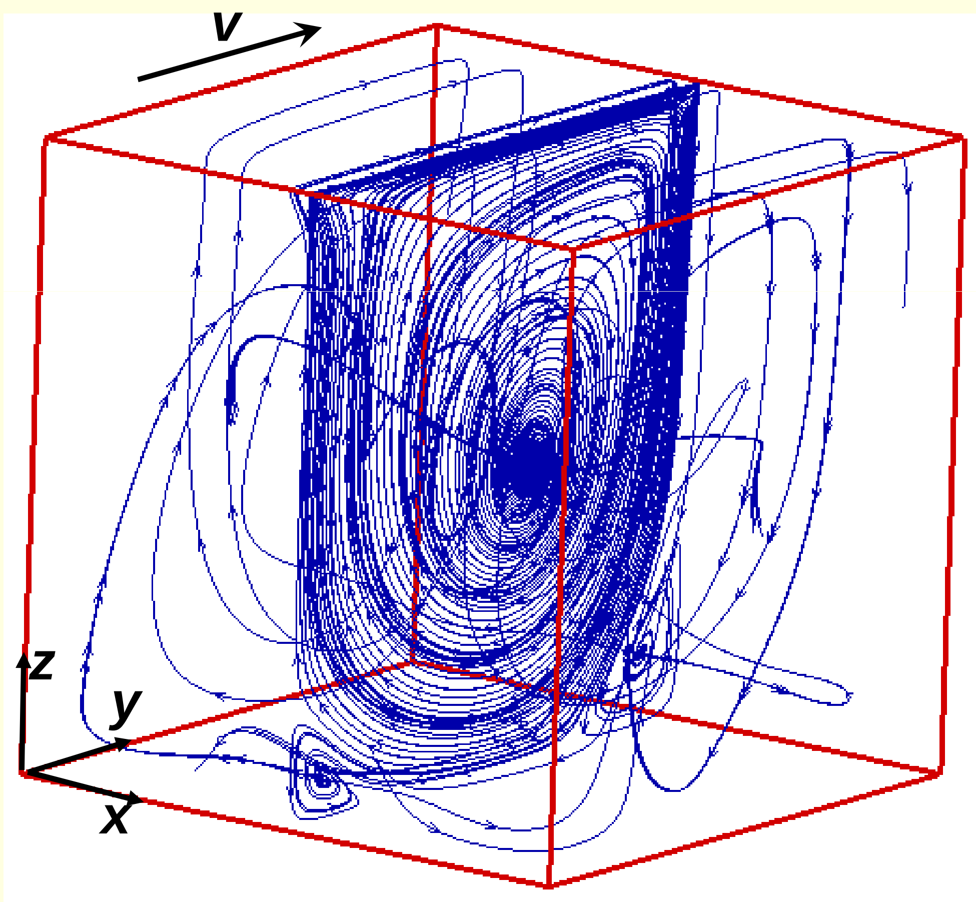


Subproject : which resolution is necessary to fit experimental data with larger Reynolds number ?

Flow visualization of cubic lid-driven cavity. steady state flow, 103^3 nodes

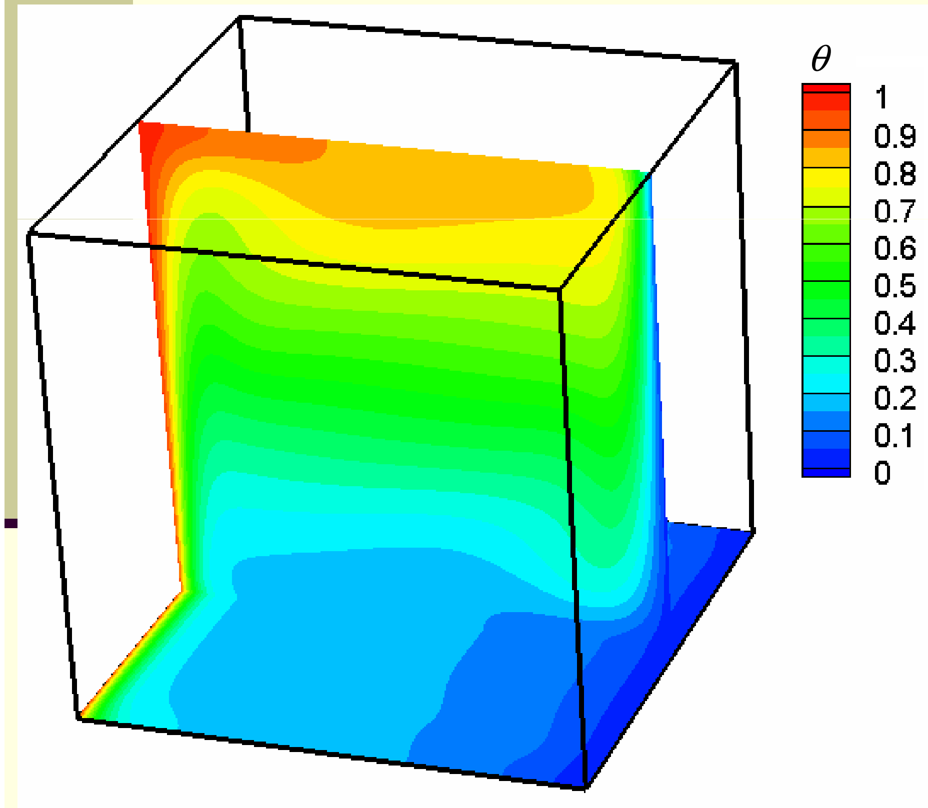
flow at Re= 10^3

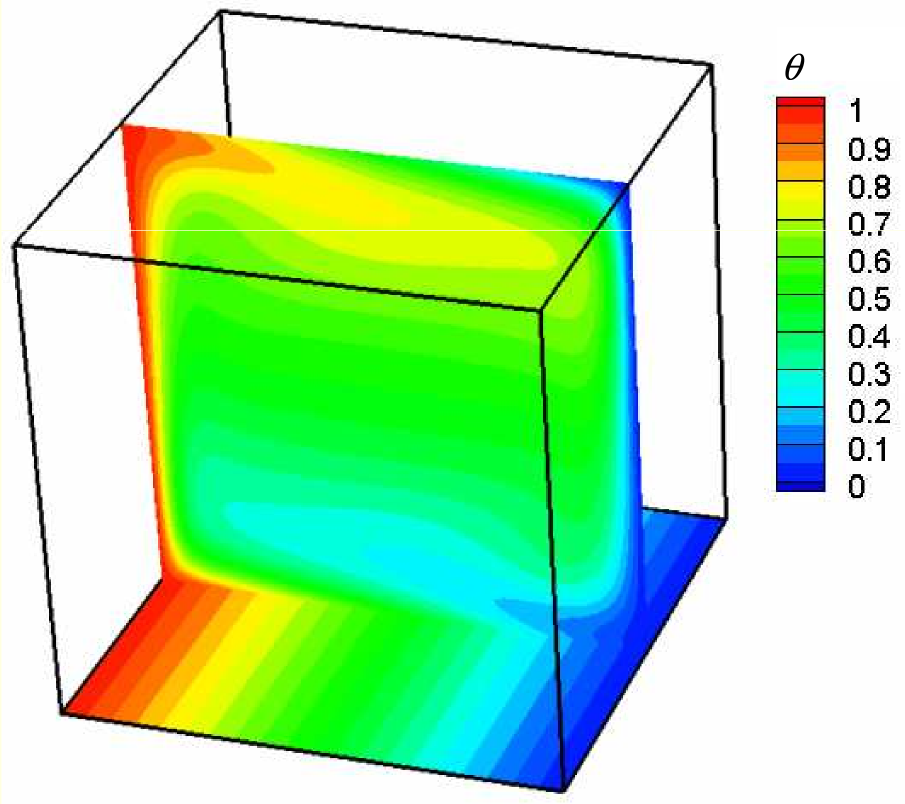




Temperature distribution in a laterally heated cubic cavity, 103^3 nodes

flow at Ra= 10^6





Conclusions

- ✓ **An Accelerated Semi-Analytical Coupled Line Implicit Gauss-Seidel Smoother (ASA-CLGS) was developed and implemented in the inner iteration of the multigrid approach.**

- ✓ **The Navier-Stokes and Boussinesq equations are solved without pressure-velocity decoupling.**

- ✓ **The code was validated against existing benchmark solutions for the lid-driven and thermally driven cavities.**

- ✓ **The approach does not require too large computational recourses allowing to perform 3D calculations on a regular PC.**

- ✓ **The characteristic CPU times consumed for a single time step per one node and per one CPU are of order 5×10^{-3} msec and 10^{-2} msec for 2D and 3D calculations, respectively.**